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CHARACTERS OF LIE GROUPS: TRACEABILITY AND CERTAIN SEMIDIRECT PRODUCTS

BY

RONALD L. LIPSMAN[†]

ABSTRACT

Two main results are obtained. First, for any unimodular type I almost connected group, it is proven that almost all of its irreducible unitary representations have global distribution characters. Second, for a certain class of semidirect products, these characters are computed and shown to be given by a function on an open dense subset, the function however not being locally integrable on the whole group.

1. Introductory comments and definitions

It has turned out to be very important for researchers doing analysis on Lie groups to have an adequate knowledge of the group's distribution characters. Indeed, the characters have played a critical role in work on the Plancherel measure, the Selberg trace formula and its applications (to automorphic forms, the topology of manifolds, etc.), and in the study of solvability of differential equations on the group and its homogeneous spaces. Thus far, character theory has been developed only for semisimple and nilpotent groups (as well as their natural generalizations, reductive and solvable groups). However, more varied kinds of Lie groups (e.g. parabolic groups and their subgroups, real algebraic groups and groups of automorphisms) have been receiving an increasing amount of attention recently. It seems appropriate that the character theory of these groups should now be more thoroughly investigated.

In this paper I will give two results in that direction. The first (Corollary 2 in Section 2) asserts that (at least unimodular type I) Lie groups have a sufficiently rich character theory. Recently, I found a similar result in [6]. However their method of proof is quite different from mine. Also theirs does not seem to

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provide for the interpretation (given in Remark 4 in Section 2) of the distribution characters as being defined on a larger space than the usual test functions.

The second result (Theorem 3 in Section 3) examines the extent to which certain non-semisimple groups can have distribution characters which are actually functions. Such a phenomenon has already been observed in some very special solvable Lie groups [1, chap. IX, §2]. In those cases certain characters turn out to be distributions which are given by functions on an open dense set; but functions which are not locally integrable on the whole group. In particular the characters are not given by functions on the *whole* group. I shall present here a class of real algebraic groups which are semidirect products of semisimple and abelian groups, and for which the same situation obtains — namely "most" of the groups characters are given by functions (on an open dense subset) that are not locally integrable on the entire group.

In a future publication I shall address myself to another topic in character theory — namely, obtaining a Kirillov-type expression for characters of general Lie groups. Some of those results as well as the results of this paper were announced in a manuscript I circulated recently [15]. I should remark that Theorem 1 was stated somewhat differently there. As it is stated here, the theorem is easier to prove (specifically it avoids some technical mumbo-jumbo on direct integrals of unbounded operators), but just as useful.

I now supply the terminology that will be used in this paper. G is a locally compact group — unimodular and type I in Section 2, real algebraic with a certain structure (to be described) in Section 3. As usual $\hat{G}$ is the set of (equivalence classes of) irreducible unitary representations. G is called almost connected if its quotient by the neutral component is compact. If $\pi \in \hat{G}$, we write $\mathcal{H}_\pi$ for a Hilbert space on which π can be realized; $\mathcal{T}(\mathcal{H}_\pi)$ denotes the trace class operators thereon; and $\|\cdot\|_t$ is the trace norm. Also $\|\cdot\|$, $\|\cdot\|_1$ stand for the usual operator norm and the L_1 norm on the group, respectively. If G is almost connected, then $\mathcal{D}(G)$ denotes the usual space of test functions with its Schwartz–Bruhat topology (see [4]). A not necessarily irreducible representation π of G is called *traceable* if for every $f \in \mathcal{D}(G)$, $\pi(f)$ is trace class and the map $\theta_\pi : f \to \operatorname{Tr} \pi(f)$ is continuous. The θ_π corresponding to irreducible representations are referred to as distribution characters. G itself is called traceable if every $\pi \in \hat{G}$ is traceable. Finally if G is unimodular and type I, then G (or $\hat{G}$) is called *traceable a.e.* if, except for a Plancherel null set, every $\pi \in \hat{G}$ is traceable.

2. Traceability a.e.

We begin this section with a statement of its main results.

THEOREM 1. *Let G be a unimodular type* I *connected Lie group with Plancherel measure μ_G. Then there exists a right-invariant differential operator D on G which has a bounded inverse satisfying: $\pi(D^{-1})$ is trace class for μ_G-almost all $\pi \in \hat{G}$.*

NOTE. The proof will reveal that D^{-1} is in the W^*-algebra generated by left translations on $L_2(G)$; as such $\pi(D^{-1})$ is well-defined for μ_G-a.a. $\pi \in \hat{G}$ (see [13, § 2]).

COROLLARY 2. *Let G be a unimodular type* I *group which is almost connected. Then with the exception of a locally μ_G-null set (a μ_G-null set, if G is separable) the representations $\pi \in \hat{G}$ are traceable.*

We shall first derive the corollary from the theorem, and then go back and prove the theorem.

PROOF OF COROLLARY 2. Here, and thoroughout the remainder of this section, G is assumed to be unimodular and type I. Assume at first also that G is a connected Lie group. By Theorem 1, we can find a right-invariant differential operator D and a μ_G-null set $\mathcal{N} \subseteq \hat{G}$ such that $\pi(D^{-1})$ is trace class for all $\pi \in \hat{G} - \mathcal{N}$. Let $\pi \in \hat{G} - \mathcal{N}$. Then for any $f \in \mathcal{D}(G)$ we have (see [13, theor. 2.1])

$$\pi(f) = \pi(D^{-1}Df) = \pi(D^{-1})\pi(Df).$$

The latter, being the product of a trace class and a bounded operator, must be trace class. More precisely, we have

$$\| \pi(f) \|_t \leqq \| \pi(D^{-1}) \|_t \, \| \pi(Df) \| \leqq \| \pi(D^{-1}) \|_t \, \| Df \|_1 .$$

Therefore if $f_n \to f$ in $\mathcal{D}(G)$, then $Df_n \to Df$ in $L_1(G)$ and so $f \to \operatorname{Tr} \pi(f)$ is continuous. Thus the representations $\pi \in \hat{G} - \mathcal{N}$ are traceable.

Next assume that G is a Lie group with finitely many connected components. Let G^0 denote the neutral component. Then G^0 is open in G. It follows that G^0 is unimodular and that $C^*(G^0)$ is naturally a C^*-subalgebra of $C^*(G)$ [17, § 1]. By [5, 4.3.5], G^0 must also be type I. Therefore by the previous case, $(G^0)^\wedge$ is traceable a.e.

Now since G^0 is of finite index in G, it must be regularly embedded. $(G^0)^\wedge$ is a Borel G-space and there is a natural map $p: \hat{G} \to (G^0)^\wedge / G$ which associates to $\pi \in \hat{G}$ its quasi-orbit in $(G^0)^\wedge$ (or what amounts to the same thing, the spectrum of $\pi|_{G^0}$). Let $\mathcal{N}_0 \subseteq (G^0)^\wedge$ be a μ_{G^0}-null set, off of which the irreducible unitary representations of G^0 are traceable. Clearly

$$\mathcal{M}_0 = G \cdot \mathcal{N}_0 = \{g \cdot \gamma : g \in G, \gamma \in \mathcal{N}_0\}$$

is a G-invariant, μ_{G^0}-null set in $(G^0)^\wedge$. Set $\mathcal{N} = p^{-1}(\mathcal{M}_0)$. By [11, theor. 10.2], $\mathcal{N}$ is a μ_G-null set in $\hat{G}$.

Now let $\pi \in \hat{G} - \mathcal{N}$. Then $\pi|_{G^0}$ is a finite direct sum of traceable representations. Hence *it* is a traceable representation of G^0. Next let $\{g_i\}_{i=1}^m$ be a set of representatives for the left cosets of G^0 in G. Then for $f \in \mathcal{D}(G)$, we have

$$\pi(f) = \int_G f(g)\pi(g)dg = \sum_{i=1}^m \int_{G_0} f(g_i g^0)\pi(g_i g^0)dg^0$$

$$= \sum_{i=1}^m \pi(g_i)\int_{G^0} f(g_i g^0)\pi(g^0)dg^0.$$

Therefore $\pi(f)$ is trace class. To show that $f \to \operatorname{Tr}\pi(f)$ is actually continuous requires a little extra reasoning. It follows from our argument in the connected case that: for any $\pi \in \hat{G} - \mathcal{N}$, there exists a differential operator D^0 on G^0 and a constant C_π (C_π depends on π while D^0 does not), such that

$$\| \sigma(h) \|_t \leqq C_\pi \| D^0 h \|_1, \quad \sigma = \pi|_{G^0}, \quad h \in \mathcal{D}(G^0).$$

Applying this inequality to the functions $f_i(g^0) = f(g_i g^0)$, we see that

$$\| \pi(f) \|_t \leqq \sum_{i=1}^m C_\pi \| D^0 f_i \|_1 .$$

It follows from this equation that π is traceable.

Finally assume only that G is almost connected. We invoke the results of [12] G is an inverse limit of Lie groups G_i (each having at most finitely many connected components); each G_i is unimodular and type I; and $(\hat{G}, \mu_G)$ is a direct limit of the $(\hat{G}_i, \mu_{G_i})$. For each i, the set $\mathcal{N}_i \subseteq \hat{G}_i$ that fails to be traceable is a μ_{G_i}-null set. It's rather simple (using [12, § 5]) to check that $\mathcal{N} = \bigcup_i \mathcal{N}_i$ is a locally μ_G-null set (actually null if the index set is countable), off of which the irreducible representations of G are traceable. This completes the proof of Corollary 2.

We turn now to Theorem 1. The idea of the proof is basically to re-adapt the beautiful results in Stinesprings' paper [18, theors. 1 and 3] to fit the formulation of non-abelian Fourier analysis found in [13]. The details follow.

PROOF OF THEOREM 1. Let Δ be the right-invariant differential operator on G given by $\Delta = X_1^2 + \cdots + X_n^2$, where $X_1, \cdots, X_n$ is a basis for $\mathfrak{g}$. Then, for sufficiently large positive integer r (e.g. $r > (1/2)\dim G$ will do), Stinesprin shows (in [18]) that $D = (1-\Delta)^r$ is an elliptic differential operator on G satisfying:

i) $(Df, f) \geqq \|f\|_2^2$, $f \in \mathcal{D}(G)$;

ii) D^{-1} has a bounded self-adjoint extension on $L_2(G)$ (also denoted D^{-1}) which commutes with right translations;

iii) there exists a continuous, square-integrable, positive-definite function φ on G such that D^{-1} agrees with left convolution by φ.

Thus, in the notation of [13], $\varphi \in L_2(G) \cap P(G)$. By Godement's factorization theorem [5, 13.8.6], there exists $\psi \in L_2(G)$ such that $\varphi = \psi * \psi$. Reasoning exactly as in [13, corol. 4.3], we conclude that $\hat{\varphi} \in L_1(\hat{G})$. Once again using the notation of [13], we then have

$$D^{-1} = \int_{\hat{G}}^{\oplus} \pi(D^{-1}) \otimes 1_\pi d\mu_G(\pi)$$

and

$$\lambda_\varphi = \int_{\hat{G}}^{\oplus} \hat{\varphi}(\pi) \otimes 1_\pi d\mu_G(\pi).$$

Since $\hat{\varphi} \in L_1(\hat{G})$, it follows that with the possible exception of a μ_G-null set, the operators

$$\hat{\varphi}(\pi) = \pi(D^{-1})$$

must in fact be trace class.

REMARKS. 1) Heretofore characters have been studied in any detail only on reductive and solvable groups. Theorem 1 and Corollary 2 show that *any* unimodular type I (almost connected) group carries *many* interesting distribution characters. What do they look like? The next section as well as a subsequent paper are addressed specifically to that problem.

2) Corollary 2 shows that unimodular type I Lie groups are "traceable a.e." Another result, which follows almost immediately from the Plancherel Theorem, is that unimodular type I groups are "CCR a.e." One wonders if the "a.e." refers to the same set in both instances. Of course traceable implies CCR, so what we are asking is whether the CCR representations are automatically traceable; or even more adventurously, whether a single CCR representation (of a connected Lie group say) must always be traceable.

3) It is interesting to speculate as to whether the following strong analog of Corollary 2 for p-adic groups might be true: Let k be a p-adic field, G the k-rational points of an algebraic group def/k, and suppose G is unimodular and type I; then $\hat{G}$ is "admissible a.e."

4) (Assume for simplicity here that G is a connected Lie group.) In the

definition of a traceable representation π we only required that $f \to \operatorname{Tr} \pi(f)$ be a continuous linear functional on $\mathcal{D}(G)$. The proof of Corollary 2 reveals that in general distribution characters have a stronger continuity property. Namely, we have the inequality

$$\| \pi(f) \|_t \leqq C_\pi \| Df \|_1,$$

for a suitable differential operator D. This implies that the map

$$f \to \pi(f), \quad \mathcal{D}(G) \to \mathcal{T}(\mathcal{H}_\pi)$$

is continuous when $\mathcal{T}(\mathcal{H}_\pi)$ has its norm topology. We see from property (i) of D that $f \to \| Df \|_1$ is also a norm on $\mathcal{D}(G)$. Let W be the completion of $\mathcal{D}(G)$ in this norm. Then $f \to \pi(f)$ extends to a norm continuous mapping from W into $\mathcal{T}(\mathcal{H}_\pi)$. The space $W_{1,2r}(G)$ of $2r$-times continuously differentiable and integrable functions on G is contained in W. I suspect, but am not certain, that $W = W_{1,2r}(G)$ and that $\| Df \|_1$ is a canonical Soboleff norm on that space. In any event I leave it to individuals especially interested in Soboleff spaces on Lie groups to find out for sure. But for future reference, we shall call the representations π for which there is $n > 0$ such that π carries $W_{1,n}(G)$ continuously into $\mathcal{T}(\mathcal{H}_\pi)$ *strongly traceable*. My last comment is that it has been realized previously that $\mathcal{D}(G)$ is not the largest space that can support all or most of the distribution characters (e.g. from [13, theor. 4.1 and corol. 4.3]) we see it might well be some larger subspace of $L_1(G) \cap A(G)$. One wonders if $W_{1,2r}(G)$ is best possible in any sense — and if not, what is?

3. Non-semisimple characters can be functions on open sets

Let $\mathbf{G}$ be a connected algebraic group with a Levi decomposition $\mathbf{G} = \mathbf{UH}$, all def/$\mathbf{R}$. Then $G = \mathbf{G}(\mathbf{R})$ is a semidirect product $G = UH$, where U is a simply connected nilpotent normal Lie subgroup and H is a reductive Lie group. One knows [16] that the irreducible representations of G are of the form $\pi = \operatorname{Ind}_{UM}^{G} \nu$, where M is the stability group in H of an irreducible representation $\gamma \in \hat{U}$ and $\nu \in (UM)^\wedge$ has the property that $\nu|_U$ is a multiple of γ.

According to the trace formula in [11, theor. 3.2], one has (at least formally and assuming G and UM are unimodular) that

$$\theta_\pi(f) = \operatorname{Tr} \pi(f) = \int_{M \backslash H} \operatorname{Tr} \int_{UM} f(h^{-1}umh)\, \nu(um)\, du\, dm\, d\bar{h}.$$

Clearly the support of θ_π couldn't contain an open set unless $M^H = \{h^{-1}mh : h \in H, m \in M\}$ contained an open subset of H. If M is reductive, that amounts to the equality rank M = rank H. That — together with known facts about characters of nilpotent groups — leads one to suspect that for θ_π to have a chance at being a function (for "generic" representations π) one needs U to be abelian and rank M = rank $H^\dagger$. Having provided that motivation, we now state the main result of this section.

THEOREM 3. *Let $G = UH$, $M = H_\gamma$, $\gamma \in \hat{U}$, be as above. Assume U is abelian and* rank M = rank H.[†] *Then $\gamma \cdot G$ is closed and M is reductive. Let $\sigma \in \hat{M}$ be such that $\pi = \mathrm{Ind}_{UM}^{G}\gamma\sigma \in \hat{G}$ is strongly traceable. Denote by H' the set of regular elements in H. Then on the open set UH' the character θ_π is a function. If C is a Cartan subgroup of H contained in M, then on UC', $C' = C \cap H'$, we have*

$$\theta_\pi(uc) = \frac{\#(W_{M,C})^{-1}}{|\Delta_{H,C}(c)|^2} \sum_{s \in W_{H,C}} |\Delta_{M,C}(c \cdot s)|^2 \gamma(u \cdot s)\,\theta_\sigma(c \cdot s), \qquad uc \in UC'.$$

Here θ_σ denotes the character of M corresponding to σ, $W_{H,C}$ and $W_{M,C}$ are the Weyl groups associated to (H, C) and (M, C), and the functions $\Delta_{H,C}, \Delta_{M,C}$ are the Jacobians that appear in the invariant integral (to be described explicitly within the proof). Finally, when $M\backslash H$ and cent $H\backslash M$ *are not compact, the function θ_π, considered as a function on G, is* not *locally integrable; so that θ_π is not a function on all of G.*

PROOF. First we demonstrate a fact needed later in the proof of Lemma 4; namely that the group $M = H_\gamma$ is actually the group of rational points of a complex algebraic group. Indeed if $\hat{U}$ is identified to $\mathfrak{u}^* = \mathrm{Hom}_{\mathbf{R}}(\mathfrak{u}, \mathbf{R})$, $\mathfrak{u} = \mathrm{LA}(U)$, via $\gamma(\exp Y) = e^{i\psi(Y)}$, $\gamma \leftrightarrow \psi \in \mathfrak{u}^* \hookrightarrow \mathbf{u}^* = \mathrm{Hom}_{\mathbf{C}}(\mathbf{u}, \mathbf{C})$, then $M = \mathbf{M}(\mathbf{R})$ where $\mathbf{M} = \mathbf{H}_\psi$. The condition rank M = rank H is the same as $\mathrm{rank}_{\mathbf{C}}\mathbf{M} = \mathrm{rank}_{\mathbf{C}}\mathbf{H}$. By [2, theor. 7.2], that forces $\psi \cdot \mathbf{H}$ to be closed in $\mathbf{u}^*$. By [3, prop. 2.3], that in turn guarantees that $\gamma \cdot H = \gamma \cdot G$ is closed in $\hat{U}$. Also since the variety $\mathbf{M}\backslash\mathbf{H} = \psi \cdot \mathbf{H}$ is affine, it follows (e.g. from [3, theor. 3.5]) that $\mathbf{M}$ (and so M) must be reductive.

Now we recall some integration formulas on real reductive algebraic groups. Let $\mathbf{H}$ be a connected reductive algebraic group def/$\mathbf{R}$, $H = \mathbf{H}(\mathbf{R})$ the group of real points, and $\mathfrak{h}$ its Lie algebra. By a Cartan subgroup C of H we mean the centralizer in H of a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{h}$, $C = Z_H(\mathfrak{c})$. The Weyl group $W_{H,C}$ is then $N_H(C)/C$ where $N_H(C)$ is the normalizer of C in H. It is a finite group of automorphisms of C. If we fix right Haar measures dh, dc on H, C, and normalize the invariant measure $d\bar{h}$ on $C\backslash H$ so that

[†] Meaning that M contains a Cartan subgroup of H.

$$\int_H f(h)dh = \int_{C\backslash H} \int_C f(ch)\,dc\,d\bar{h}, \quad f \in \mathcal{D}(H),$$

then we have the Harish–Chandra–Weyl invariant integral formula

$$\int_{H'_C} f(h)dh = w_{H,C}^{-1} \int_C \int_{C\backslash H} |\Delta_{H,C}(c)|^2 f(h^{-1}ch)\,d\bar{h}\,dc, \quad f \in \mathcal{D}(H').$$

Here $w_{H,C} = \#(W_{H,C})$, $H'_C = \{h^{-1}ch : h \in H,\ c \in C'\}$ and

$$\Delta_{H,C}(c) = \xi_\rho(c) \prod_{\alpha \in Q} (1 - \xi_\alpha(h^{-1})),$$

where Q denotes a set of positive roots for $(\mathfrak{h}, \mathfrak{c})$,

$$\rho = \tfrac{1}{2} \sum_{\alpha \in Q} \alpha,$$

and ξ_ρ, ξ_α denote as usual the lifts to C of the linear forms ρ, α. Recall also that H' is a disjoint union of the open sets H'_C as C varies over a set of representatives for the non-conjugate Cartan subgroups.

Now let C be a Cartan subgroup of H which is contained in M. It is easy to check that C is also a Cartan subgroup of M. Fix right Haar measures dh, dm, dc. Choosing as usual invariant measures $d\bar{h}, d\bar{m}$ on $C\backslash H$, $C\backslash M$ we have the formulas

$$\int_{H'_C} f(h)dh = w_{H,C}^{-1} \int_C \int_{C\backslash H} |\Delta_{H,C}(c)|^2 f(h^{-1}ch)\,d\bar{h}\,dc, \quad f \in \mathcal{D}(H')$$

$$\int_{M''_C} f(m)dm = w_{M,C}^{-1} \int_C \int_{C\backslash M} |\Delta_{M,C}(c)|^2 f(m^{-1}cm)\,d\bar{m}\,dc, \quad f \in \mathcal{D}(M'').$$

Here prime (′) denotes regularity in H, while double prime (″) denotes regularity in M. It is clear that elements of M which are H-regular must also be M-regular, i.e. $M \cap H' \subseteq M''$. It is also clear that there is a unique choice of invariant measure $d\bar{g}$ on $UM\backslash G$ so that

$$\int_{C\backslash H} f(h)\,d\bar{h} = \int_{UM\backslash G} \int_{C\backslash M} f(mg)\,d\bar{m}\,d\bar{g},$$

for $f \in C^\infty(G)$, left UC-invariant, and compactly supported mod UC. Note that we have not assumed G is unimodular. If we denote by δ the modulus for the action of H on U.

$$\delta(h) \int_U f(huh^{-1})du = \int_U f(u)du, \; h \in H, \quad f \in \mathcal{D}(U),$$

then the modular function of G is $\delta_G(g) = \delta_G(uh) = \delta(h)$.

Let $\pi = \mathrm{Ind}_{UM}^{G}\gamma\sigma$ be as postulated in the statement of the theorem. Let $f \in \mathcal{D}(G)$ be of the form $f = \varphi * \varphi^*$, $\varphi \in C_0(G)$. Then by [11, theor. 3.2], we have

$$\mathrm{Tr}\,\pi(f) = \int_{UM\backslash G} \delta(g)^{-1}\,\mathrm{Tr}\int_{UM} f(g^{-1}umg)\gamma(u)\sigma(m)\,dudmd\bar{g}.$$

But by strong traceablility and [1, p. 251], we can conclude that this formula holds for all $f \in \mathcal{D}(G)$. We proceed to compute for $f \in \mathcal{D}(UH'_C)$:

$$\mathrm{Tr}\,\pi(f) = \int_{UM\backslash G} \delta(g)^{-1}\mathrm{Tr}\int_{UM} f(g^{-1}umg)\gamma(u)\sigma(m)\,dudmd\bar{g}$$

$$(1)\qquad = \int_{UM\backslash G} \delta(g)^{-1}\int_{UM} f(g^{-1}umg)\gamma(u)\theta_\sigma(m)\,dudmd\bar{g}$$

$$= w_{M,C}^{-1}\int_{UM\backslash G} \delta(g)^{-1}\int_{UC}\int_{C\backslash M} |\Delta_{M,C}(c)|^2 f(g^{-1}um^{-1}cmg) \times \gamma(u)\theta_\sigma(c)\,dudcd\bar{m}d\bar{g}$$

$$(2)\qquad = w_{M,C}^{-1}\int_{UM\backslash G} \delta(g)^{-1}\int_{UC}\int_{C\backslash M} |\Delta_{M,C}(c)|^2 f(g^{-1}m^{-1}ucmg)\delta(m)^{-1} \times \gamma(u)\theta_\sigma(c)\,dudcd\bar{m}d\bar{g}$$

$$(\mathrm{I})\qquad = w_{M,C}^{-1}\int_{C\backslash H}\int_{UC} \delta(h)^{-1}|\Delta_{M,C}(c)|^2 f(h^{-1}uch)\gamma(u)\theta_\sigma(c)\,dudcd\bar{h}.$$

We used in step (1) the fact that for $g \in G$, $m \to \int_U f(g^{-1}umg)\gamma(u)du$ is a test function on M, and in step (2) that M fixes γ.

We now define a class function on $G = UH$. Consider first

$$\theta_C : uc \to w_M^{-1}\Big(\sum_{s\in W_H} \gamma(u\cdot s)\theta_\sigma(c\cdot s)|\Delta_M(c\cdot s)|^2\Big)|\Delta_H(c)|^{-2},$$

for $uc \in UC'$. (Here W_H, w_m, Δ_H, etc. are abbreviations for $W_{H,C}$, $w_{M,C}$, $\Delta_{H,C}$, etc.) Note that $\gamma(u\cdot s)$ is well-defined because C stabilizes γ. Suppose there is $g \in G$ such that $gucg^{-1} = u_1c_1 \in UC'$. Write $g = vh \in UH$. Then

$$\begin{aligned} u_1c_1 &= vhuch^{-1}v^{-1} \\ &= v(huh^{-1})(hch^{-1}v^{-1}hc^{-1}h^{-1})hch^{-1}. \end{aligned}$$

Therefore $c_1 = hch^{-1}$, which implies $h \in N_H(C)$. Also

$$u_1 = v(huh^{-1})(c_1v^{-1}c_1^{-1}).$$

Then

$$\theta_C(u_1c_1) = w_M^{-1}\left(\sum \gamma(u_1\cdot s)\,\theta_\sigma(c_1\cdot s)|\Delta_M(c_1\cdot s)|^2\right)|\Delta_H(c_1)|^{-2}$$
$$= w_M^{-1}\left(\sum \gamma\cdot s^{-1}(vhuh^{-1}c_1v^{-1}c_1^{-1})\,\theta_\sigma(c\cdot hs)|\Delta_M(c\cdot hs)|^2\right)|\Delta_H(c)|^{-2}.$$

But since C fixes γ, we have

$$\gamma\cdot s^{-1}(v)\,\gamma\cdot s^{-1}(huh^{-1})\,\gamma\cdot s^{-1}(c_1v^{-1}c_1^{-1}) = \gamma\cdot s^{-1}(huh^{-1}).$$

Continuing the computation, we get

$$\theta_C(u_1c_1) = w_M^{-1}\left(\sum \gamma(u\cdot hs)\,\theta_\sigma(c\cdot hs)|\Delta_M(c\cdot hs)|^2\right)|\Delta_H(c)|^{-2}$$
$$= \theta_C(uc),$$

after transforming $s\to h^{-1}s$. Thus if we define

$$\theta_C(g^{-1}ucg) = \theta_C(uc),$$

we get a G-class function on the open set UH'_C.

Now it is a relatively simple matter, using the nature of the map

$$C\backslash H\times C'\to H'_C$$
$$(\bar h, c)\to h^{-1}ch$$

(consult e.g. [8, § 20]), to see that θ_C is a continuous function on UH'_C. Hence if we are given a function $f\in\mathcal{D}(UH'_C)$, the integral

$$\int_{UH'_C} f(g)\,\theta_C(g)\,dg$$

is absolutely convergent.

We now compute for $f\in\mathcal{D}(UH'_C)$:

$$\int_{UH'_C} f(g)\,\theta_C(g)\,dg = \int_{UH'_C} f(uh)\,\theta_C(uh)\,du\,dh$$
$$= w_H^{-1}\int_{UC}\int_{C\backslash H} f(uh^{-1}ch)\theta_C(uh^{-1}ch)|\Delta_H(c)|^2\,du\,dc\,d\bar h$$
$$= w_H^{-1}\int_{UC}\int_{C\backslash H} |\Delta_H(c)|^2 f(h^{-1}uch)\theta_C(h^{-1}uch)\delta(h)^{-1}\,du\,dc\,d\bar h$$
$$= w_H^{-1}w_M^{-1}\int_{UC}\int_{C\backslash H} |\Delta_H(c)|^2 f(h^{-1}uch)\left(\Sigma_{w_H}\gamma(u\cdot s)\theta_\sigma(c\cdot s)\right.$$
$$\left.\times|\Delta_M(c\cdot s)|^2\right)|\Delta_H(c)|^{-2}\delta(h)^{-1}\,du\,dc\,d\bar h$$

$$(3) \qquad = w_H^{-1} w_M^{-1} \sum \int_{UC} \int_{C\backslash H} f(h^{-1}s^{-1}ucsh)\gamma(u)\theta_\sigma(c)|\Delta_M(c)|^2 \times \delta(sh)^{-1} du\, dc\, d\bar{h}$$

$$(4) \qquad = w_H^{-1} w_M^{-1} \sum \int_{UC} \int_{C\backslash H} f(h^{-1}uch)\gamma(u)\theta_\sigma(c)|\Delta_M(c)|^2 \times \delta(h)^{-1} du\, dc\, d\bar{h}$$

$$(\mathrm{II}) \qquad = w_M^{-1} \int_{UC} \int_{C\backslash H} f(h^{-1}uch)\gamma(u)\theta_\sigma(c)|\Delta_M(c)|^2 \delta(h)^{-1} du\, dc\, d\bar{h}.$$

In step (3) we transformed $u \to u \cdot s^{-1}$, $c \to c \cdot s^{-1}$, and in step (4) we used the H-invariance of $d\bar{h}$ and the unimodularity of H.

Comparing (I) and (II) we conclude that for $f \in \mathcal{D}(UH'_C)$,

$$\mathrm{Tr}\, \pi(f) = \int f\theta_C.$$

Next we shall establish an auxiliary lemma.

LEMMA 4. *Let C be a Cartan subgroup of H. Then either C is H-conjugate to another Cartan subgroup which is actually contained in M, or $C' \cap M^H = \varnothing$.*

Combining our previous computations with this lemma, we will have determined the character of π completely on the open dense set UH'.

PROOF OF LEMMA 4. Suppose $c \in C' \cap M^H$. Then $c \in C'$ and there is $h \in H$ such that $hch^{-1} \in M$. Since c is regular, hch^{-1} is also H-regular, and so M-regular. It follows that the centralizer C_1 in M of hch^{-1} is a Cartan subgroup of M, $C_1 = Z_M(hch^{-1})$. Suppose we knew that $C_1 = Z_H(hch^{-1})$. Then $C_1 = Z_M(hch^{-1}) = Z_H(hch^{-1}) = hCh^{-1}$ and we would be done. Let $\mathfrak{c}_1 = LA(C_1)$. Then $\mathfrak{c}_1$ is a Cartan subalgebra of $\mathfrak{m}$, and so by rank also of $\mathfrak{h}$. Let $D = Z_H(\mathfrak{c}_1)$. Obviously $C_1 \subseteq D$; what must be shown is that $C_1 = D$. Now we may find a Cartan subalgebra $\boldsymbol{\mathfrak{c}}_1$ of $\boldsymbol{\mathfrak{m}}$ such that $\mathfrak{c}_1 = \boldsymbol{\mathfrak{c}}_1 \cap \mathfrak{m}$. Then $\mathbf{D} = Z_{\mathbf{H}}(\boldsymbol{\mathfrak{c}}_1)$ is a (Zariski) connected complex Cartan subgroup of $\mathbf{H}$. But $LA(\mathbf{D} \cap \mathbf{M}) = \boldsymbol{\mathfrak{c}}_1 \cap \boldsymbol{\mathfrak{m}} = \boldsymbol{\mathfrak{c}}_1$. Therefore $(\mathbf{D} \cap \mathbf{M})^0 = \mathbf{D}$, and so $\mathbf{D} = (\mathbf{D} \cap \mathbf{M})^0 \subseteq \mathbf{D} \cap \mathbf{M} \subseteq \mathbf{D}$.

It follows that $\mathbf{D} \subseteq \mathbf{M}$. Now one easily checks that $D = Z_H(\mathfrak{c}_1) = \mathbf{D}(\mathbf{R})$. Therefore

$$C_1 = Z_M(\mathfrak{c}_1) = M \cap \mathbf{D} = (H \cap \mathbf{M}) \cap \mathbf{D}$$
$$= H \cap (\mathbf{M} \cap \mathbf{D}) = H \cap \mathbf{D} = D.$$

It remains finally to show that the character is not a locally integrable function on the whole group. It is clearly enough to show $\theta|_H$ is not locally integrable on

H in a neighborhood of 1. First, let C be a Cartan subgroup of M and H, $C \subseteq M \subseteq H$. Since $\operatorname{Cent} H \backslash M$ is not compact, C may be chosen so that $C/\operatorname{Cent} H$ is not compact. Then a maximal compact subgroup K of H can be chosen so that C is invariant under the corresponding Cartan involution. Moreover we can find a (cuspidal) parabolic subgroup of H and a Langland's decomposition $P = LAN$ so that C is a fundamental Cartan subgroup of $\Xi = La$ (i.e. $A \backslash C$ is compact).

Now let $f \in \mathscr{D}(G)$ be non-negative and invariant under inner automorphisms of K. Then

$$\int_H f|\theta| \geqq \int_{H_C^{\cdot}} f|\theta|$$

$$= w_H^{-1} \int_C \int_{C\backslash H} f(h^{-1}ch)|\theta(c)|\Delta_H(c)|^2 d\bar{h}dc.$$

We apply [9, corol. 2, p. 94] to obtain the equation

$$|\Delta_+(c)|^{1/2} \int_{C\backslash H} f(h^{-1}ch)d\bar{h} = \delta_p(c)^{1/2} \int_{C\backslash \Xi} \int_N f(\xi^{-1}cn\xi)dnd\bar{\xi}, \quad c \in C', \tag{5}$$

where

$$\Delta_+(c) = \det(1 - Ad(c))|_{\mathfrak{h}/\mathfrak{b}}, \quad \mathfrak{b} = LA(L).$$

Now let $\mathscr{U}$ be a neighborhood of 1 in G on which $f(x) \equiv 1$. Then there must exist a neighborhood $\mathscr{V}$ of 1 in C and a positive constant ε such that the right side of (5) is larger than ε whenever $c \in C' \cap \mathscr{V}$. Thus it is enough to prove that $|\theta||\Delta_H|^2|\Delta_+|^{-1/2}$ is not locally integrable on C around 1. From an examination of θ_C we see that is tantamount to showing $|\Delta_+|^{-1/2}|\Delta_M|$ is not locally integrable on C around 1. Reducing in the usual way to the Lie algebra $\mathfrak{h}$, we find we must show non-local integrability on $\mathfrak{h}$ around 0 of the function

$$F = \frac{\prod_{\alpha \in \mathscr{S}} \alpha}{\prod_{\alpha \in \mathscr{T}} \alpha},$$

where $\mathscr{S} =$ the positive roots of $(\mathfrak{m}, \mathfrak{c})$ and $\mathscr{T} =$ the positive roots of $(\mathfrak{h}, \mathfrak{c})$ which are not identically zero on the split part $\mathfrak{a}$ of $\mathfrak{c}$. The point is that by the non-compactness of $M\backslash H$, we can find $\alpha \in \mathscr{T} - \mathscr{S}$. This can be seen as follows: First it is enough to assume $\mathfrak{h}$ is simple. Then one checks easily that the orthocomplement of $\mathfrak{m}$ (wrt the Killing form) in $\mathfrak{h}$ actually generates $\mathfrak{h}$. Thus the set of roots of $(\mathfrak{h}, \mathfrak{c})$ which are not roots of $(\mathfrak{m}, \mathfrak{c})$ spans the set of all roots of $(\mathfrak{h}, \mathfrak{c})$. On the other hand $\mathscr{T} \subseteq \mathscr{S}$ would say that every root of $(\mathfrak{h}, \mathfrak{c})$ which is not a root of $(\mathfrak{m}, \mathfrak{c})$ must vanish on $\mathfrak{a}$. By the preceding remark on the span, all roots of $(\mathfrak{h}, \mathfrak{c})$ would then vanish on $\mathfrak{a}$. This can happen only if $\mathfrak{a} \equiv \{0\}$. That contradicts the fact that $C/\text{Cent } H$ is non-compact.

Thus letting $\alpha \in \mathcal{T} - \mathcal{S}$, we can find a compact neighborhood Ω of 0 in $\mathfrak{h}$ and a positive constant k such that

$$|F(Y)| \geqq k\,|\alpha(Y)|^{-m}, \quad Y \in \Omega,$$

where $m = 1$ or 2 according as α takes only real values on $\mathfrak{c}$ or not. Our result follows finally from the fact that neither dt/t nor $dxdy/(x^2 + y^2)$ is locally integrable around 0 in $\mathbf{R}$ or $\mathbf{R}^2$ respectively.

REMARKS. 1) The proof of non-local integrability for the special case that M is a maximally split Cartan subgroup of H was communicated to me in a letter by Harish-Chandra. The general proof given above is a natural generalization of this argument. I wish to thank Joe Wolf for pointing out to me why $\mathcal{T} - \mathcal{S}$ must be non-empty, and for helping me over another small, but annoying blind spot I encountered while writing down the proof.

2) In the case that M contains only compact (mod Cent H) Cartan subgroups of H, that is, when $\text{Cent}\, H \backslash M$ is compact, I have not been able to settle whether the character is locally integrable or not.

3) According to [14, theor. 3.4], there is a Zariski-open subset $\mathcal{O}$ of $\hat{U}$ on which the function $\gamma \to \{\text{conjugacy class of } H_\gamma\}$ is finite-valued. In many instances these H_γ are reductive (e.g. if the generic orbits are closed). If in addition G is unimodular, then it follows from Corollary 2 and [11, theor. 10.2] that $\pi = \text{Ind}_{UH_\gamma}^{G}\, \gamma\sigma$ is strongly traceable for "most" $\gamma \in \mathcal{O}$ and $\sigma \in \hat{H}_\gamma$. (I suspect that *all* the irreducible representations sitting over a closed orbit are at least traceable, but I have not proven it.) In case rank $H_\gamma = \text{rank}\, H$, the distribution character θ_π is completely determined by Theorem 3 on the open set UH'. Unfortunately, the proof gives no insight as to what θ_π looks like on the singular set $U(H - H')$. I hope to supply some in the next paper.

We close by citing some examples.

EXAMPLE. 1) Let $G = UH$, $U = \mathfrak{h} = LA(H)$, with H acting on U by the adjoint action. If B denotes the Killing form on $\mathfrak{h}$, the characters of U are given by $\gamma_v(u) = e^{iB(u,v)}$, $u, v \in \mathfrak{h}$. Since H leaves B invariant, we have $H_{\gamma_v} = H_v$. The conditions of Theorem 3 will be satisfied, for example if v is a regular element of $\mathfrak{h}$. In that case $M = H_v$ is a Cartan subgroup of H. It was this example that inspired Theorem 3.

2) For an odd integer $n > 1$, let $\mathbf{U} = \mathbf{C}^n$ carry a non-degenerate quadratic form

q (with integral coefficients). If $\mathbf{H}$ denotes the subgroup of $SL(n, \mathbf{C})$ that preserves q, then $\mathbf{H} \cong SO(n, \mathbf{C})$ and $\mathbf{G} = \mathbf{UH}$ is an algebraic group. If $G = UH$ is a real form and $\gamma_v(u) = e^{iq(u,v)}$, then $H_{\gamma_v} = H_v$. When v is restricted to those elements for which $q(v, v) \neq 0$, the conditions of Theorem 3 are fulfilled. In this case the group $M = H_v$ is a real form of $SO(n-1, \mathbf{C})$ which depends on the signature of $q|_U$ and the sign of $q(v, v)$.

3) More examples — in which $\mathbf{H} = \mathrm{Sp}(n, \mathbf{C})$ or certain exceptional groups — can be gleaned from the tables in [7].

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University of Maryland

and

The Hebrew University of Jerusalem
Jerusalem, Israel